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M-Splines

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1. INTRODUCTION

In the study of generalized splines there has been a continuing search for those properties which are most essential. Schoenberg [12], Greville [6], Ahlberg, Nilson and Walsh [1], Schultz and Varga [13], Schultz [14] and Lucas [10] have pursued an explicit approach which defines generalized splines as functions which are in the null space of a self-adjoint differential operator except at given grid points where additional continuity or interpolation requirements are imposed. DeBoor and Lynch [4], Atteia [3], Anselone and Laurent [2], Golomb [5], Jerome and Schumaker [8], and Jerome and Varga [9] have developed an implicit approach which defines generalized splines as those elements in a Hilbert space X which minimize a bilinear functional of the form $M(g, g) = (Tg, Tg)_Y$ over translates of the null space, $N(\Lambda)$, of an associated family of continuous linear functionals Λ. Here, T is a continuous linear mapping of the Hilbert space X onto a Hilbert space Y, whose null space is finite dimensional. A consequence of the latter approach is that a function s is a spline if and only if s satisfies the orthogonality condition

$$M(s, g) = 0 \qquad \text{for all} \quad g \in N(\Lambda) \tag{1.1}$$

[8, Theorem 2.1].

It is the purpose of this paper to develop the consequences of beginning a study of generalized splines, herein denoted by M-splines, by taking the orthogonality condition (1.1) as their defining characteristic in place of the earlier minimization condition. This approach will be more general than any of those considered in the earlier quoted papers, and as Example 1 will show, actually includes most of the spline characterizations of each of these papers as special cases. A new class of spline functions related to a continuous bilinear functional M which is not necessarily symmetric will be introduced in Example 2. These splines include the generalized L-splines of Schultz [14] and Lucas [10].

2. Preliminaries

The following formalization of the notion of an M-spline includes the generalized splines of Anselone and Laurent [2], Golomb [5], and Jerome and Schumaker [8].

Definition 1. Let X be a real Hilbert space, and Λ a family of continuous linear functionals over X. Associate with Λ the linear space $N(\Lambda) = \{n \in X: \lambda(n) = 0 \text{ for all } \lambda \in \Lambda\}$, which we shall refer to as the null space of Λ. Let $M(x, y)$ be a continuous bilinear functional on $X \times X$ such that $M(n, n) \geqslant 0$ for all $n \in N(\Lambda)$. A function $s \in X$ is said to be an *M-spline* if $M(s, n) = 0$ for all $n \in N(\Lambda)$. The class of all M-splines for a fixed Λ is denoted by $\mathrm{Sp}(M, \Lambda)$.

Definition 2. Let X, Λ and M be as above, and let $x \in X$. Then any $s \in X$ is said to be a Λ-interpolate of x if $s - x \in N(\Lambda)$. If s is also in $\mathrm{Sp}(M, \Lambda)$, then s is said to be an *$\mathrm{Sp}(M, \Lambda)$-interpolate of x*.

Note that s is a Λ-interpolate of x if and only if $\lambda(s) = \lambda(x)$ for all $\lambda \in \Lambda$. Also observe that $\mathrm{Sp}(M, \Lambda)$ is a closed linear space.

In the next section we shall give conditions which insure the existence of an $\mathrm{Sp}(M, \Lambda)$-interpolate of any element in X. If for a given X, Λ and M, with $M(n, n) \geqslant 0$ for all $n \in N(\Lambda)$, as in Definition 1, we define N_1 by $N_1 = \{n_1 \in N(\Lambda): M(n_1, n_1) = 0\}$, then it may easily be seen that N_1 is a closed linear subspace of X:

N_1 is clearly homogeneous. If $x, y \in N_1$, let $\alpha = M(x, y) + M(y, x)$. Then $M(x - \alpha y, x - \alpha y) = -\alpha[M(x, y) + M(y, x)] = -\alpha^2 \geqslant 0$ since $x - \alpha y \in N(\Lambda)$. Therefore $\alpha = M(x, y) + M(y, x) = 0$ for all $x, y \in N_1$. Thus $M(x + y, x + y) = 0$ for any $x, y \in N_1$, and N_1 is additive. By the continuity of M, N_1 is closed.

Definition 3. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$ such that

$$M(n, n) \geqslant 0 \qquad \text{for all} \quad n \in N(\Lambda). \tag{2.1}$$

If there is an $m > 0$ such that

$$M(n, n) \geqslant m \| n \|^2 \qquad \text{for all} \quad n \in N(\Lambda), \tag{2.2}$$

then we shall say that *the system* $\{X, \Lambda, M, N(\Lambda)\}$ *is well-posed.* Denote by N_1 the closed linear subspace of $N(\Lambda)$,

$$N_1 = \{n_1 \in N(\Lambda): M(n_1, n_1) = 0\}. \tag{2.3}$$

If

$$M(x, n_1) = 0 \qquad \text{for all} \quad x \in X,\ n_1 \in N_1, \tag{2.4}$$

and if there exists a closed linear subspace of $N(\Lambda)$, N_2, such that

$$N(\Lambda) = N_1 \oplus N_2, \tag{2.5}$$

and an $m > 0$ such that

$$M(n_2, n_2) \geqslant m \| n_2 \|^2 \qquad \text{for all} \quad n_2 \in N_2, \tag{2.6}$$

then we shall say that *the system* $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ *is* N_1*-posed.*

Note that if $N_1 = \{0\}$, $\{X, \Lambda, M, N(\Lambda)\}$ is well-posed if and only if $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is $\{0\}$-posed.

EXAMPLE 1. Let X and Y be real Hilbert spaces, Λ a family of continuous linear functionals on X, and T a continuous linear transformation of X onto Y, such that the dimension of the null space of T, $N(T)$, is finite. Define the continuous bilinear functional M by

$$M(x_1, x_2) = (Tx_1, Tx_2)_Y \qquad \text{for all} \quad x_1, x_2 \in X.$$

Then $M(x, x) \geqslant 0$ for all $x \in X$, and $N_1 = N(\Lambda) \cap N(T)$. If $n \in N_1$, $M(x, n) = (Tx, Tn)_Y = 0$ since $n \in N(T)$, so (2.4) is satisfied. Let $N_2 = (N_1)^{\perp}_{N(\Lambda)}$, the orthogonal complement of N_1 in $N(\Lambda)$. Since N_2 is closed and $N(T)$ is finite dimensional, $N_2 + N(T)$ is closed [7, Prob. 8], and by Lemma 2.1 developed by Golomb and Jerome [5], $T(N_2)$ is closed. Thus T maps N_2 $1-1$ onto the closed subspace $T(N_2)$, and therefore T restricted to N_2 has a continuous inverse by the open mapping theorem. Thus there is an $m > 0$ such that $\| Tn_2 \| \geqslant m \| n_2 \|$ for all $n_2 \in N_2$. But then $M(n_2, n_2) = \| Tn_2 \|_Y^2 \geqslant m^2 \| n_2 \|^2$ for all $n_2 \in N_2$ giving (2.6), and thus the system $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed. If $N_1 = N(\Lambda) \cap N(T) = \{0\}$ then $N(\Lambda) = N_2$, and the system $\{X, \Lambda, M, N(\Lambda)\}$ is well-posed.

3. EXISTENCE AND UNIQUENESS OF M-SPLINES

The following theorem gives conditions which insure the existence and uniqueness of M-splines. Note that there is no symmetry requirement placed on M.

THEOREM 1. *Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. If the system $\{X, \Lambda, M, N(\Lambda)\}$ is well-posed then for any $y \in X$ there is a unique* Sp(M, Λ)-*interpolate s of y, which depends continuously on y.*

Proof. Since M is continuous, there is a $K > 0$ such that

$$K \| x \| \| n \| \geqslant M(x, n) \qquad \text{for all} \quad x \in X, n \in N(\Lambda). \tag{3.1}$$

Thus for any fixed $x \in X$, $M(x, -)$ is a bounded linear functional on $N(\Lambda)$. Therefore, there is a $z \in N(\Lambda)$ such that $M(x, n) = (z, n)$ for all $n \in N(\Lambda)$. Let $Tx = z$. Then T is a continuous linear mapping of X into $N(\Lambda)$, such that

$$K \| n \|^2 \geqslant M(n, n) = (Tn, n) \geqslant m \| n \|^2 \qquad \text{for all} \quad n \in N(\Lambda). \tag{3.2}$$

Denote by T_N the restriction of T to $N(\Lambda)$. Clearly T_N is $1 - 1$. It will now be shown that the range of T_N, $R(T_N)$, is actually equal to $N(\Lambda)$. Suppose $\{n_i\}_{i=0}^{\infty} \in R(T_N)$ and $n_i \to n \in N(\Lambda)$. Then there exist $x_i \in N(\Lambda)$ such that $Tx_i = n_i$. From (3.2),

$$\| Tn \| \geqslant m \| n \| \qquad \text{for all } n \in N(\Lambda). \tag{3.3}$$

Since $\{n_i\}$ is a Cauchy sequence, so is $\{Tx_i\}$. But by (3.3), $\{x_i\}$ must then be Cauchy also. Let $x_i \to x \in N(\Lambda)$. Then since T is continuous, $Tx_i \to Tx$, so $Tx = n$. This establishes that $R(T_N)$ is closed. Let n_1 be in the orthogonal complement of $R(T_N)$ in $N(\Lambda)$. Then

$$0 = (Tn_1, n_1) \geqslant m \| n_1 \|^2.$$

Therefore, $\| n_1 \| = 0$, so $n_1 = 0$, and $R(T_N) = N(\Lambda)$. Since T_N is a $1 - 1$ mapping of $N(\Lambda)$ onto $N(\Lambda)$, by the open mapping theorem T_N has a continuous inverse T_N^{-1}.

Now let $y \in X$. Suppose $s \in \mathrm{Sp}(M, \Lambda)$ and $s = y + \bar{n}$ with $\bar{n} \in N(\Lambda)$. Then $M(y + \bar{n}, n) = 0$ for all $n \in N(\Lambda)$. Therefore $T(y + \bar{n}) = 0$, implying that $\bar{n} = -T_N^{-1}(Ty)$. Thus

$$s = (I - T_N^{-1} T) y \in y + N(\Lambda). \tag{3.4}$$

So if there is an Sp(M, Λ)-interplate of y, s, then s is unique, and is given as a continuous function of y by (3.4). But for any $n \in N(\Lambda)$, $M(s, n) = (Ts, n) = (Ty - Ty, n) = 0$, so (3.4) actually gives an Sp(M, Λ)-interpolate of y, establishing the theorem.

COROLLARY 1. *Under the conditions of Theorem 1,*

$$X = N(\Lambda) \oplus \mathrm{Sp}(M, \Lambda).$$

Proof. If $y \in X$, $y = (y - s) + s$, where s is the *unique* Sp(M, Λ)-interpolate of y.

COROLLARY 2. *Under the conditions of Theorem* 1, *if span* (Λ) *has a basis of dimension n, then dim*(Sp(M, Λ)) $= n$.

COROLLARY 3 (Anselone and Laurent [2]). *In Example* 1, *if*

$$N(T) \cap N(\Lambda) = \{0\},$$

then for every $x \in X$, *there is a unique* Sp(M, Λ)-*interpolate* s *which depends continuously on* x.

The following example introduces a class of nonselfadjoint generalized splines which include the generalized L-splines of Schultz [14] and Lucas [10] as special cases.

EXAMPLE 2. Let X be the Sobolev space $W^{n,2}[a, b]$ of all functions f in $C^{n-1}[a, b]$ whose $n - 1$st derivative is absolutely continuous and $D^n f \in L^2[a, b]$, with inner product

$$(f, g)_n = \sum_{i=0}^{n} \int_a^b [D^i f(t)][D^i g(t)]\, dt.$$

Define a continuous bilinear functional M on $X \times X$ by

$$M(f, g) = \sum_{i,j=0}^{n} \int_a^b b_{ij}(t)[D^i f(t)][D^j g(t)]\, dt,$$

where $b_{nn}(t) \geqslant \omega$, $a \leqslant t \leqslant b$ for some $\omega > 0$, and where the b_{ij} are bounded, real-valued measurable functions on $[a, b]$, $0 \leqslant i, j \leqslant n$. Suppose Λ is a family of continuous linear functionals over X which includes functionals of the type $\lambda(f) = f(x), x \in [a, b]$, for all $f \in X$. Denote the set of $x \in [a, b]$ for which there is such a λ by Δ and let $\bar{\Delta}$ be the greatest distance between the points into which $[a, b]$ is thus partitioned. It is shown in [11] that there exist positive constants ϵ and m such that if $\bar{\Delta} < \epsilon$, $M(u, u) \geqslant m \| u \|_n^2$ for all $u \in N(\Lambda)$. Thus the system $\{W^{n,2}[a, b], \Lambda, M, N(\Lambda)\}$ is well-posed for any such Λ, and by the previous theorem for any function $f \in W^{n,2}[a, b]$ there is a unique Sp(M, Λ)-interpolate which depends continuously on f. Some properties and applications of these nonselfadjoint generalized splines are developed in [11]. For their numerical utilization it is necessary to have some characterization results. These have also been developed.

The next theorem separates the questions of existence and uniqueness of M-splines, generalizing Theorem 1.

THEOREM 2. *Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. Suppose there is a closed subspace of $N(\Lambda)$, N_2, such that the system $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed, where N_1 is defined by* (2.3). *Then for any $y \in X$ there is a unique* $\mathrm{Sp}(M, \Lambda)$*-interpolate s of y in $y + N_2$ which depends continuously on y. Moreover, any other interpolate of y, $\bar{s}$, is an* $\mathrm{Sp}(M, \Lambda)$*-interpolate of y if and only if $\bar{s} - s \in N_1$.*

Proof. By hypothesis there is a closed subspace N_2 of $N(\Lambda)$ such that (2.5) and (2.6) are valid. Let Λ_2 be the orthogonal complement of N_2 in X. Then Λ_2 can be considered as a family of continuous linear functionals on X whose null space, $N(\Lambda_2)$, is N_2. Thus, by (2.6) the system $\{X, \Lambda_2, M, N(\Lambda_2)\}$ is well-posed (where $N(\Lambda_2) = N_2$); so by Theorem 1 for every $y \in X$ there is a unique $\mathrm{Sp}(M, \Lambda_2)$-interpolate s of y which depends continuously on y. This gives a unique $\hat{n}_2 \in N_2$ such that $s = y + \hat{n}_2$ and $M(s, n_2) = 0$ for all $n_2 \in N_2$. By (2.4) $M(s, n_1) = 0$ for all $n_1 \in N_1$, and by (2.5) any $n \in N(\Lambda)$ is of the form $n = n_1 + n_2$ with $n_1 \in N_1$, $n_2 \in N_2$. Therefore s is a unique $\mathrm{Sp}(M, \Lambda)$-interpolate of y in $y + N_2$.

Next it will be established that

$$M(n_1, n) = 0 \qquad \text{for all} \quad n_1 \in N_1, n \in N(\Lambda). \tag{3.5}$$

Let $n_1 \in N_1$, $n = \bar{n}_1 + \bar{n}_2 \in N(\Lambda)$ with $\bar{n}_1 \in N_1$, $\bar{n}_2 \in N_2$. Then by (2.4), $M(n_1, n) = M(n_1, \bar{n}_2)$. Consider for *any* real α, $M(\bar{n}_2 + \alpha n_1, \bar{n}_2 + \alpha n_1) = M(\bar{n}_2, \bar{n}_2) + \alpha M(n_1, \bar{n}_2) \geqslant 0$ by (2.4), (2.3) and (2.1). Then $M(n_1, \bar{n}_2)$ must be zero, or the above inequality could not hold for *all* α, establishing (3.5).

Now if s is the unique $\mathrm{Sp}(M, \Lambda)$-interpolate of y in $y + N_2$, and $\bar{s}$ is any other $\mathrm{Sp}(M, \Lambda)$-interpolate of y, then $\bar{s} - s \in N(\Lambda)$, and $M(\bar{s} - s, n) = 0$ for all $n \in N(\Lambda)$. Letting $n = \bar{s} - s$, we see by (2.3) that $\bar{s} - s \in N_1$. On the other hand, if s is as above and $\bar{s} - s \in N_1$, then $\bar{s} = s + \bar{n}_1$ for some $\bar{n}_1 \in N_1$ and $M(\bar{s}, n) = M(s, n) + M(\bar{n}_1, n) = 0$ for all $n \in N(\Lambda)$ by (3.5) and Definition 1, so $\bar{s}$ is an $\mathrm{Sp}(M, \Lambda)$-interpolate of y.

COROLLARY 4. *Under the conditions of Theorem* 2,

$$X = N_2 \oplus \mathrm{Sp}(M, \Lambda).$$

COROLLARY 5. *Under the conditions of Theorem* 2, *if* $\mathrm{span}(\Lambda)$ *has a basis of dimension r, and* $\dim(N_1) = r_1$, *then* $\dim(\mathrm{Sp}(M, \Lambda)) = r + r_1$.

The following corollary shows that if M is symmetric and nonnegative over all of X, the orthogonality condition (2.4) is always satisfied, giving again the conclusions of Theorem 2.

COROLLARY 6. *Let X be a real Hilbert space, Λ a family of continuous linear* *functionals on X, and M a continuous symmetric bilinear functional on $X \times X$ such that $M(x, x) \geqslant 0$ for all $x \in X$. Let $N_1 = \{n_1 \in N(\Lambda)\colon M(n_1, n_1) = 0\}$ and suppose that there is some closed subspace of $N(\Lambda)$, N_2, such that $N(\Lambda) = N_1 \oplus N_2$ and M is positive definite on N_2. Then for any $y \in X$, there is a unique* Sp(M, Λ)*-interpolate s of y in $y + N_2$, which depends continuously on y. Moreover, any other interpolate of y, $\bar{s}$, is an* Sp(M, Λ)*-interpolate of y if and only if $\bar{s} - s \in N_1$.*

Proof. Except for the orthogonality condition (2.4), the system $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed. But (2.4) does hold, since for any $x \in X, n_1 \in N_1$, $M(x + \alpha n_1, x + \alpha n_1) = M(x, x) + 2\alpha M(x, n_1) \geqslant 0$ for all real α, implying that $M(x, n_1) = 0$.

COROLLARY 7 (Golomb [5], Jerome and Schumaker [8]). *In Example 1, for any $x \in X$ there exists an* Sp(M, Λ)*-interpolate s, and any other interpolate of x, $\bar{s}$, is an* Sp(M, Λ)*-interpolate of x if and only if $\bar{s} - s \in N(\Lambda) \cap N(T)$.*

4. SUFFICIENT CONDITIONS FOR WELL-POSED AND N_1-POSED SYSTEMS

The next result gives a very useful condition which insures the existence of the space N_2 of Definition 3 and Theorem 2.

THEOREM 3. *Let X, Λ, M and N_1 be given as in Definition* 3 *with M and N_1 satisfying* (2.1), (2.3) *and* (2.4). *Suppose there is a closed subspace of $N(\Lambda)$, N_3, such that $N_1 + N_3$ is of finite codimension in $N(\Lambda)$, and a $m_1 > 0$ such that*

$$M(n_3, n_3) \geqslant m_1 \| n_3 \|^2 \qquad \text{for all} \quad n_3 \in N_3. \tag{4.1}$$

Then there exists a closed subspace of $N(\Lambda)$, N_2, containing N_3 such that the system $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed.

Proof. Since the codimension of $N_1 + N_3$ in $N(\Lambda)$ is finite, and $N_1 \cap N_3 = \{0\}$ by (2.3) and (4.1),

$$N(\Lambda) = N_1 \oplus N_3 \oplus N_4 \tag{4.2}$$

for some finite dimensional subspace of $N(\Lambda)$, N_4. It will now be shown that $N_2 = N_3 \oplus N_4$ satisfies (2.6) as well as (2.5), demonstrating that the system $(X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed. It will suffice to show this for the case where N_4 is one dimensional.

Suppose N_4 consists of the span of some $n_4 \in N(\Lambda) - (N_1 \oplus N_3)$. Let Λ_3 be the orthogonal complement of N_3 in X. Then Λ_3 can be considered to be a family of continuous linear functionals on X, with null space $N(\Lambda_3) = N_3$. Now let us define a continuous bilinear functional on $X \times X$, M_s, by

$$M_s(x, y) = \tfrac{1}{2}[M(x, y) + M(y, x)] \qquad \text{for all} \quad x, y \in X. \tag{4.3}$$

By (4.1) and (4.3),

$$M_s(n_3, n_3) = M(n_3, n_3) \geqslant m_1 \| n_3 \|^2 \quad \text{for all } n_3 \in N(\Lambda_3) = N_3. \tag{4.4}$$

By (4.4) the system $\{X, \Lambda_3, M_s, N(\Lambda_3)\}$ is well-posed, so by Theorem 1, there is a unique $\mathrm{Sp}(M_s, \Lambda_3)$-interpolate of n_4, $\bar{s} \in n_4 + N_3$, satisfying

$$2M_s(\bar{s}, n_3) = M(\bar{s}, n_3) + M(n_3, \bar{s}) = 0 \qquad \text{for all} \quad n_3 \in N_3. \tag{4.5}$$

Let $m = \frac{1}{2}\min(M(\bar{s}, \bar{s})/\| \bar{s} \|^2, m_1)$. Then

$$\begin{aligned} M(\alpha\bar{s} + n_3, \alpha\bar{s} + n_3) &= \alpha^2 M(\bar{s}, \bar{s}) + \alpha[M(\bar{s}, n_3) + M(n_3, \bar{s})] + M(n_3, n_3) \\ &\geqslant 2m(\alpha^2 \| \bar{s} \|^2 + \| n_3 \|^2) \qquad (4.6) \\ &\geqslant m \| \alpha\bar{s} + n_3 \|^2, \end{aligned}$$

by use of (4.5) and the parallelogram inequality. But $\mathrm{span}\{n_4 + N_3\} = \mathrm{span}\{\bar{s} + N_3\}$, so (4.6) establishes that M is positive definite on $N_4 \oplus N_3$. If $\dim(N_4) > 1$, the above argument may be repeated.

In the literature [9, 13] one usually finds one set of hypotheses for the unique existence of spline interpolates and an additional one, usually in the form of the mesh norm being sufficiently small, for error bounds. Theorem 3 can be used [11] to show that the second requirement in all such cases is redundant and that in fact unique existence implies that error bounds hold. This will be treated elsewhere.

The next example utilizes Theorem 3 to demonstrate the existence of nonselfadjoint splines which are not unique.

EXAMPLE 3. Let $X = W_0^{1,2}[0, 1] \cap S$ where S is the set of all functions defined on [0, 1] which are symmetric about the line $x = 1/2$ and $W_0^{1,2}[0, 1]$ is the subset of all $u \in W_0^{1,2}[0, 1]$ such that $u(0) = u(1) = 0$. Let Λ consist just of the linear functional λ, where

$$\lambda(f) = f(1/6) - f(1/2) + f(5/6).$$

and let

$$M(u, v) = \int_0^1 u'v' - \pi u'v + \pi uv' - \pi^2 uv \, dt.$$

Then $M(u, u) = \int_0^1 (u')^2 - \pi^2 u^2 \, dt \geqslant 0$ for all $u \in X$ by the Rayleigh Ritz inequality with $M(u, u) = 0$ if and only if u is a multiple of $\sin \pi t$. Clearly, since $\lambda(\sin \pi t) = 0$, $N_1 = \text{span}\{\sin \pi t\}$. A straightforward integration by parts shows that $M(u, \sin \pi t) = 0$ for all $U \in X$, so (2.4) holds in addition to (2.1) and (2.3). For the purpose of applying Theorem 3, let $\lambda_1(u) = u(1/2)$ for all $u \in X$ and $N_3 = \{u \in N(\Lambda): \lambda_1(u) = 0\}$. Note that $N_1 + N_3$ is of codimension 1 in $N(\Lambda)$. Another application of the Rayleigh Ritz inequality shows that there is a positive m such that $M(u, u) \geqslant m \| u \|^2$ for all $u \in N_3$. Hence by Theorem 3 there exists a closed subspace of X, N_2, such that $X = N_1 \oplus N_2$ and the system $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed. Moreover, by Theorem 2 for every $u \in X$ there exists a unique $\text{Sp}(M, \Lambda)$-interpolate s of u in $u + N_2$ which depends continuously on u and any other interpolate of u, $\bar{s}$, is an $\text{Sp}(M, \Lambda)$-interpolate of u if and only if $\bar{s} = s + \alpha \sin \pi t$ for some real number α.

THEOREM 4. *Let X be a real Hilbert spae, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. Suppose there is a closed subspace of $N(\Lambda)$, N_2, such that the system $\{X, \Lambda, M, N(\Lambda), N_1, N_2\}$ is N_1-posed. If $\Lambda_1 \supset \Lambda$ is another family of continuous linear functionals on X, and if the codimension* (codim) *of $N(\Lambda_1)$ in $N(\Lambda)$ is finite, then there exists a closed subspace $N_2^{(1)}$ such that the system $\{X, \Lambda_1, M, N(\Lambda_1), N_1^{(1)}, N_2^{(1)}\}$ is $N_1^{(1)}$-posed.*

Proof. Let $N_1^- = N_1 \cap N(\Lambda_1)$ and $N_3 = N_2 \cap N(\Lambda_1)$. Then with $N_1^{(1)} = \{n \in N(\Lambda_1): M(n, n) = 0\}$, $N_1^{(1)} = N_1^-$. Since $\text{codim}(N(\Lambda_1))$ in $N(\Lambda)$ is finite it follows that $\text{codim}(N_1^{(1)})$ in N_1 is finite and $\text{codim}(N_3)$ in N_2 is finite. Thus since $N(\Lambda) = N_1 \oplus N_2$, and $N_1^{(1)} \subset N_1$, $N_3 \subset N_2$, $N_1^{(1)} + N_3$ must be of finite codimension in $N(\Lambda_1)$, and M is positive definite on N_3. Therefore by Theorem 3, there exists a closed subspace of $N(\Lambda_1)$, $N_2^{(1)}$, containing N_3 such that the system $\{X, \Lambda_1, M, N(\Lambda_1), N_1^{(1)}, N_2^{(1)}\}$ is $N_1^{(1)}$-posed.

COROLLARY 8. *Let X be a real Hilbert space, $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals such that $\Lambda_{i+1} \supset \Lambda_i$ and* $\text{codim}(N(\Lambda_{i+1}))$ *in $N(\Lambda_i)$ is finite for $i \geqslant 0$, and let M be a continuous bilinear functional on $X \times X$ such that there is a closed subspace of $N(\Lambda_0)$, $N_2^{(0)}$, such that the system $\{X, \Lambda_0, M, N(\Lambda_0), N_1^{(0)}, N_2^{(0)}\}$ is $N_1^{(0)}$-posed. Then for all $i \geqslant 0$ there is a closed subspace of $N(\Lambda_i)$, $N_2^{(i)}$, such that the system $\{X, \Lambda_i, M, N(\Lambda_i), N_1^{(i)}, N_2^{(i)}\}$ is $N_1^{(i)}$-posed, where*

$$N_1^{(i)} = \{n \in N(\Lambda_i) : M(n, n) = 0\}.$$

Moreover, if for any $i_0 \geqslant 0$, $N_1^{(i_0)} = \{0\}$, then for all $i \geqslant i_0$, the system $\{X, \Lambda_i, M, N(\Lambda_i)\}$ is well-posed, and no restriction need be placed on $\text{codim}(N(\Lambda_{i+1}))$ *in $N(\Lambda_{i_0})$.*

An important application of this corollary is the situation where the system

$\{X, \Lambda_0, M, N(\Lambda_0), N_1^{(0)}, N_2^{(0)}\}$ is $N_2^{(0)}$-posed and Λ_{i+1} is formed from Λ_i by augmenting Λ_i with one continuous linear functional not in the span of Λ_i, $i \geqslant 0$. Then the system $\{X, \Lambda_i, M, N(\Lambda_i), N_1^{(i)}, N_2^{(i)}\}$ is $N_1^{(i)}$-posed for all $i \geqslant 0$. Note that in this application it does not matter whether or not the dimension of the span of Λ_0 is finite, and if $N_1^{(i_0)} = \{0\}$, then Λ_{i+1} may be formed from Λ_i by augmenting Λ_i with any set of continuous linear functionals, for all $i \geqslant i_0$.

5. Extremal Results

For a given real Hilbert space X and continuous bilinear functional on $X \times X$, M, it will be useful to associate another continuous bilinear functional on $X \times X$, M_s, defined by

$$M_s(x, y) = \tfrac{1}{2}[M(x, y) + M(y, x)] \qquad \text{for all} \quad x, y \in X. \tag{5.1}$$

Then $M_s(x, x) = M(x, x)$ for all $x \in X$, and if M is symmetric, $M_s(x, y) = M(x, y)$ for all $x, y \in X$. The following lemma generalizes a result usually referred to as the "first integral relation" [13, Theorem 4].

Lemma 1. *Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$ such that $M(n, n) \geqslant 0$ for all $n \in N(\Lambda)$. Then for any $x \in X$ if s is an* $\mathrm{Sp}(M_s, \Lambda)$*-interpolate of x,*

$$M(x, x) = M(x - s, x - s) + M(s, s). \tag{5.2}$$

Proof. Since s is an $\mathrm{Sp}(M_s, \Lambda)$-interpolate of x, $M_s(s, x - s) = 0$, so

$$\begin{aligned} M(x, x) &= M(x - s, x - s) + 2M_s(s, x - s) + M(s, s) \\ &= M(x - s, x - s) + M(s, s). \end{aligned}$$

The next theorem gives an extremal result for M-splines generalizing [10, Theorem 7].

Theorem 5. *Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. Suppose $M(n, n) \geqslant 0$ for all $n \in N(\Lambda)$, and that there is a closed subspace N_2 of $N(\Lambda)$ such that the system $\{X, \Lambda, M_s, N(\Lambda), N_1, N_2\}$ is N_1-posed where*

$$N_1 = \{n_1 \in N(\Lambda) \colon M_s(n_1, n_1) = 0\}.$$

Then for any $y \in X$;

(i) *there is at least one* $\mathrm{Sp}(M_s, \Lambda)$*-interpolate* s *of* y,

(ii) $M(s, s) = \min_{x \in X}\{M(x, x)$: *$x$ is a Λ-interpolate of y*$\}$, (5.3)

(iii) *if x is a Λ-interpolate of y, and* $M(x, x) = M(s, s)$,

then $x - s \in N_1$ *and x is also an* $\mathrm{Sp}(M_s, \Lambda)$*-interpolate of* y.

Proof. Property i follows immediately from Theorem 2. Suppose x is any Λ-interpolate of y. Then s is also an $\mathrm{Sp}(M_s, \Lambda)$-interpolate of x, and since $x - s \in N(\Lambda)$, $M(x - s, x - s) \geqslant 0$ and (5.2) of the lemma implies that $M(x, x) \geqslant M(s, s)$ giving property (ii). If, in addition, $M(x, x) = M(s, s)$, then (5.2) of the lemma implies that $M_s(x - s, x - s) = 0$, so $x - s \in N_1$, and by Theorem 2, $x \in \mathrm{Sp}(M_s, \Lambda)$.

COROLLARY 9. *Under the hypothesis of Theorem* 5, *if* $N_1 = \{0\}$ *then the* $\mathrm{Sp}(M_s, \Lambda)$*-interpolate, s, of y gives the* unique *solution to the extremal problem* (5.3).

COROLLARY 10. *Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on* $X \times X$ *such that* $M(x, x) \geqslant 0$ *for all* $x \in X$. *Suppose there are closed subspaces of* $N(\Lambda)$, N_1 *and* N_2, *such that* $N(\Lambda) = N_1 \oplus N_2$, $M(n_1, n_1) = 0$ *for all* $n_1 \in N_1$, *and* $M(n_2, n_2) \geqslant m \| n_2 \|^2$ *for all* $n_2 \in N_2$. *Then for any* $y \in X$, *there is at least one* $\mathrm{Sp}(M_s, \Lambda)$*-interpolate s of y, and the extremal problem* (5.3) *is solved by s. Moreover, if x is any other Λ-interpolate of y which minimizes M as in* (5.3), *then* $x \in \mathrm{Sp}(M_s, \Lambda)$ *and* $x - s \in N_1$.

Proof. Equations (2.1), (2,3), (2.5) and (2.6) are explicitly satisfied by M_s, N_1 and N_2. Just as in the proof of Corollary 6,

$$M(x, x) = M_s(x, x) \geqslant 0 \qquad \text{for all } x \in X \text{ implies that } M_s(x, n_1) = 0$$

for all $x \in X$, $n_1 \in N_1$. Therefore the orthogonality condition (2.4) also holds and the system $\{X, \Lambda, M_s, N(\Lambda), N_1, N_2\}$ is N_1-posed. Thus Theorem 5 applies.

Theorem 5 and Corollary 9 are useful in identifying M-splines with other spline characterizations. For instance in Example 2 let M be given by

$$M(u, v) = \sum_{i=0}^{n} \int_a^b a_i(t)[D^i u(t)][D^i v(t)]\, dt,$$

let Λ be such that the system $\{W^{n,2}[a, b], \Lambda, M, N(\Lambda)\}$ is well-posed, and let s be the unique $\mathrm{Sp}(M, \Lambda)$-interpolate of $f \in W^{n,2}[a, b]$. If Λ consists solely of the Hermite type functionals considered in [10], then Theorem 7 of [10] asserts that the element minimizing (5.3) is the unique generalized *L-spline* interpolate $\tilde{s}$ of f, and hence $\tilde{s} = s$ by Corollary 9. Similar arguments show that the γ-elliptic splines of Schultz [14], the R-splines of Golomb [5] and the Lg-splines of Jerome and Schumaker [8] are also special cases of M-splines.

The next result generalizes Theorem 6 of [10], and also implicitly offers a generalization of the property P used in that paper.

THEOREM 6. *Let X be a real Hilbert space and $\{\Lambda_i : i \geqslant 0\}$ be a nested sequence of families of continuous linear functionals on X, such that $\Lambda_{i+1} \supset \Lambda_i$ and the codimension of $N(\Lambda_{i+1})$ in $N(\Lambda_i)$ is finite, for all $i \geqslant 0$. Suppose M is a continuous bilinear functional on $X \times X$ such that $M(x, x) \geqslant 0$ for all $x \in N(\Lambda_0)$, and the system $\{X, \Lambda_0, M_s, N(\Lambda_0), N_1^{(0)}, N_2^{(0)}\}$ is N_1-posed. Then for any $x \in X$, and all $i \geqslant 0$;*

(i) *the system $\{X, \Lambda_i, M_s, N(\Lambda_i), N_1^{(i)}, N_2^{(i)}\}$ is $N_1^{(i)}$-posed,*

(ii) *there is at least one* $\mathrm{Sp}(M_s, \Lambda_i)$*-interpolate s_i of x,*

(iii) $M(x - s_i, x - s_i) = \min\{M(x - \bar{s}, x - \bar{s})$:

$\bar{s} \in \mathrm{Sp}(M_s, \Lambda_i) \cap U\}$, *where $U = \{y: y$ is a Λ_0-interpolate of $x\}$.*

If $M(y, y) \geqslant 0$ for all $y \in X$, then

(iv) $M(x - s_i, x - s_i) = \min\{M(x - \bar{s}, x - \bar{s}): \bar{s} \in \mathrm{Sp}(M_s, \Lambda_i)\}$.

Proof. Property i follows immediately from Corollary 8, and property (ii) follows from property (i) and Theorem 2. Suppose $\bar{s} \in \mathrm{Sp}(M_s, \Lambda_i)$. Then $s_i - \bar{s}$ is an $\mathrm{Sp}(M_s, \Lambda_i)$-interpolate of $x - \bar{s}$, so substituting $x - \bar{s}$ for x and $s_i - \bar{s}$ for s in (5.2) of Lemma 1 gives

$$M(x - \bar{s}, x - \bar{s}) = M(x - s_i, x - s_i) + M(s_i - \bar{s}, s_i - \bar{s}). \quad (5.4)$$

If s is also in U, then $s_i - \bar{s} \in N(\Lambda_0)$ so $M(s_i - \bar{s}, s_i - \bar{s}) \geqslant 0$, and from (5.4), $M(x - \bar{s}, x - \bar{s}) \geqslant M(x - s_i, x - s_i)$ establishing property (iii). Similarly, if $M(y, y) \geqslant 0$ for all $y \in X$, then again $M(s_i - \bar{s}, s_i - \bar{s}) \geqslant 0$ giving property (iv) from (5.4).

6. CONVERGENCE OF M-SPLINES

The next theorem generalizes a result of Golomb [5, Corollary 7.1].

THEOREM 7. *Let X be a real Hilbert space, $\{\Lambda_i\}_{i=0}^{\infty}$ a nested sequence of families of continuous linear functionals on X with $\Lambda_{i+1} \supset \Lambda_i$, $i \geqslant 0$, and*

M a continuous symmetric bilinear functional on $X \times X$. Suppose the system $\{X, \Lambda_0, M, N(\Lambda_0)\}$ is well-posed, and let $\Lambda_\infty = \bigcup_{i=0}^{\infty} \Lambda_i$. Then for all $i \geq 0$ and every $x \in X$, there is a unique $\text{Sp}(M, \Lambda_i)$*-interpolate s_i of x, and a unique* $\text{Sp}(M, \Lambda_\infty)$*-interpolate of x, s_∞, where $\{s_i\}$ and s_∞ depend continuously on x. Moreover,* $\lim_{i\to\infty} s_i = s_\infty$ *and* $\lim_{i\to\infty} M(s_i, s_i) = M(s_\infty, s_\infty)$.

Proof. The system $\{X, \Lambda_0, M, N(\Lambda_0)\}$ is well-posed, so there exists an $m > 0$ such that

$$M(n, n) \geq m \| n \|^2 \qquad \text{for all} \quad n \in N(\Lambda_0). \tag{6.1}$$

Hence (6.1) also holds for all $n \in N(\Lambda_\infty)$ and all $n \in N(\Lambda_i)$, and thus the systems $\{X, \Lambda_\infty, M, N(\Lambda_\infty)\}$ and $\{X, \Lambda_i, M, N(\Lambda_i)\}$ for all $i \geq 0$ are well-posed, and the first part of the theorem follows from Theorem 1. Since s_∞ is a Λ_j-interpolate of $s_j \in \text{Sp}(M, \Lambda_j)$ for all $j \geq 0$, and if $j \geq i \geq 0$, s_j is a Λ_i-interpolate of $s_i \in \text{Sp}(M, \Lambda_i)$, property (ii) of Theorem 5 gives

$$M(s_\infty, s_\infty) \geq M(s_j, s_j) \geq M(s_i, s_i) \qquad \text{for all} \quad j \geq i \geq 0. \tag{6.2}$$

Lemma 1 then gives,

$$M(s_j - s_i, s_j - s_i) = M(s_j, s_j) - M(s_i, s_i), \tag{6.3}$$

since s_i is an $\text{Sp}(M, \Lambda_i)$-interpolate of s_j. From (6.2) and (6.3) it follows that $\lim_{i,j\to\infty} M(s_j - s_i, s_j - s_i) = 0$, and by (6.1), $\{s_i\}_{i=0}^{\infty}$ is a Cauchy sequence. Let $s^\infty = \lim_{i\to\infty} s_i$. Then it may easily be seen that $s_\infty = s^\infty$, and the result follows.

As an application of this theorem, consider a variation of Example 2 with $\{\Lambda_i\}_{i=0}^{\infty}$ being a nested sequence of families of continuous linear functionals on $W^{n,2}[a, b]$, $n \geq 1$, and $\{\bar{\Delta}_i\}_{i=0}^{\infty}$ the associated partition norms. Suppose $\bar{\Delta}_i \to 0$. Then there is an $i_0 \geq 0$ such that the system $\{W^{n,2}[a, b], \Lambda_{i_0}, M, N(\Lambda_{i_0})\}$ is well-posed, and for every $f \in W^{n,2}[a, b]$ there is a unique $\text{Sp}(M, \Lambda_i)$-interpolate s_i for all $i \geq i_0$. Let s_∞ be the $\text{Sp}(M, \bigcup_{i=0}^{\infty} \Lambda_i)$-interpolate of f, and let $x \in [a, b]$. Since $\bar{\Delta}_i \to 0$, there is a sequence $\{x_i\}$ with $x_i \in \Delta_i$ such that $x_i \to x$. But $f(x_i) = s_i(x_i) = s_\infty(x_i)$, so $f(x) = s_\infty(x)$ by the continuity of f and s_∞. Therefore $f \equiv s_\infty$, and by Theorem 7,

$$\lim_{i\to\infty} \| f - s_i \|_n = 0. \tag{6.4}$$

The convergence in the Sobolev norm given by (6.4) may be easily shown to imply uniform convergence of $D^j s_i$ to $D^j f$ on $[a, b]$ for $0 \leq j \leq n - 1$ and L^2 convergence when $j = n$.

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